

General Solution Example 3

Example 4: Find the general solution of the following linear system in *vector form*.

$$\begin{bmatrix} 1 & -4 & 5 & 1 \\ 0 & 1 & -1 & 4 \\ -1 & 4 & -5 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} \quad (1)$$

$$\begin{aligned} & \left[\begin{array}{cccc|c} 1 & -4 & 5 & 1 & 2 \\ 0 & 1 & -1 & 4 & 4 \\ -1 & 4 & -5 & 0 & -1 \end{array} \right] \xrightarrow{R_3 := R_3 + R_1} \left[\begin{array}{cccc|c} 1 & -4 & 5 & 1 & 2 \\ 0 & 1 & -1 & 4 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \xrightarrow{\substack{R_1 := R_1 - R_3 \\ R_2 := R_2 - 4R_3}} \left[\begin{array}{cccc|c} 1 & -4 & 5 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \\ & \xrightarrow{R_1 := R_1 + 4R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \xrightarrow{\substack{x_4 = 1 \\ x_3 = t}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \end{aligned}$$

$$x_2 - x_3 = 0 \Rightarrow x_2 = x_3 = t$$

$$x_1 + t = 1 \Rightarrow x_1 = 1 - t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1-t \\ t \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -t \\ t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$